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Stability of Hot Reissner–Nordstrom Black Hole

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The stability of de Sitter space in the presence of a black hole is shown. The gravitational and electromagnetic perturbations of a Reissner–Nordstrom black hole which is asymptotically de Sitter instead of asymptotically flat are considered in terms of complex potentials. This result on stability can also be applied to the inflationary scenario.

1. INTRODUCTION

The problem of finding perturbations of solutions to the Einstein field equations has been considered in recent years. The perturbations of the Reissner–Nordstrom (RN) black hole which is asymptotically flat have been studied by several authors (Moncrief, 1974*a*,*b*, 1975; Zerilli, 1974; Lee, 1979; Wald, 1979; Chandrasekhar, 1983; Torres del Castillo, 1987). Most of the studies have been made either directly (by considering the metric perturbations) or through the perturbations of Newman–Penrose quantities. In both approaches, however, to find the complete perturbations requires rather lengthy and by no means straightforward computations (Chandrasekhar, 1983). A third procedure to obtain the simultaneous gravitational electromagnetic perturbations of the RN black hole has been presented recently by Torres del Castillo (1987), who obtained the complete perturbations in terms of the derivatives of two complex potentials.

In this paper we study the complete perturbations of the RN black hole which is asymptotically de Sitter instead of asymptotically flat, using the method of complex potentials. The reason behind choosing this type of model is that there has been renewed interest in the cosmological constant because of its presence in the inflationary scenario of the early universe. We could term this type of RN black hole a hot RN black hole, since the

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de Sitter spacetime has been interpreted as hot vacuum spacetime (Gasperini, 1988).

In Section 2, we give the description of the hot RN solution and the basic equations of our problem in terms of the Geroch–Held–Penrose (GHP) formalism (Geroch *et al.*, 1973) and finally obtain a set of four linear ordinary differential equations. In Section 3, we transform these equations into a one-dimensional Schrödinger-type wave equation and discuss the stability of the spacetime. This result on stability can also be applied to the inflationary scenario (Guth, 1981).

2. HOT RN SOLUTION AND THE BASIC EQUATIONS

The hot RN metric is given by

$$ds^2 = e^{2\nu}\, dt^2 - e^{-2\nu}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{1}$$

Here

$$e^{2\nu} = \frac{Y}{r^2}$$

$$Y = r^2 - \alpha r^4 - 2mr + e^2, \qquad \alpha = \frac{\Lambda}{3}$$

where Λ is the cosmological constant and m and e are the mass and the electric charge parameters. We take the null tetrads whose (t, r, θ, ϕ) components are given by

$$\begin{aligned} l^\mu &= \left(\frac{r^2}{Y}, 1, 0, 0\right) \\ n^\mu &= \left(\frac{1}{2}, -\frac{1}{2}\frac{Y}{r^2}, 0, 0\right) \\ m^\mu &= \left(0, 0, \frac{1}{\sqrt{2}\, r}, \frac{i}{\sqrt{2}\, r \sin\theta}\right) \end{aligned} \tag{2}$$

The nonvanishing spin coefficients are as follows:

$$\begin{aligned} \rho &= -\frac{1}{r}, \qquad \beta = \frac{1}{2\sqrt{2}}\frac{\cot\theta}{r} = \beta' \\ \rho' &= \frac{Y}{2r^3}, \qquad \varepsilon' = \rho' - \frac{\dot{Y}}{4r^2} \end{aligned} \tag{3}$$

where dot denotes differentiation with respect to the arguments. The components of the electromagnetic field are given by

$$\phi_0 = \phi_2 = 0, \qquad \phi_1 = e/2r^2 \tag{4}$$

The only nonvanishing components of the Weyl and the Ricci tensors are given by

$$\psi_2 = -\frac{m}{r^3} + \frac{e^2}{r^4}, \qquad \phi_{11} = \frac{e^2}{2r^4} \tag{5}$$

If η is a quantity of type $\{p, q\}$ with a time dependence of the form $e^{i\omega t}$, one finds from (2) and (3) that the effect of the GHP operators on η is given by

$$\begin{aligned} þ\eta &= \left(\partial_r + i\omega \frac{r^2}{Y}\right)\eta \\ þ'\eta &= -\frac{Y^{1-\sigma}}{2r^2}\left(\partial_r - i\omega \frac{r^2}{Y} - \frac{2\sigma}{r}\right) Y^{\sigma}\eta \\ ð\eta &= \frac{\sin^s\theta}{\sqrt{2}\, r}(\partial_\theta + i \operatorname{cosec}\theta\, \partial_\phi) \sin^{-s}\theta\; \eta \\ ð'\eta &= \frac{\sin^{-s}\theta}{\sqrt{2}\, r}(\partial_\theta - i \operatorname{cosec}\theta\, \partial_\phi) \sin^{s}\theta\; \eta \end{aligned} \tag{6}$$

where $\sigma = (p+q)/2$ and $s = (p-q)/2$ are the boost weight and the spin weight of η, respectively.

As is obtained in Torres del Castillo (1987) the metric and the vector potential perturbations of a solution to the Einstein–Maxwell equations in terms of two complex potentials in a frame such that ϕ_1 is the only nonvanishing component of the background electromagnetic field and $\kappa = \sigma = 0$, can be obtained as

$$\begin{aligned} h_{\mu\nu} = 2\{l_\mu l_\nu[(ð - \tau)M_{\dot 2} + \bar\sigma' M_{\dot 1}] + m_\mu m_\nu(þ - \rho)M_{\dot 1} \\ - l_{(\mu} m_{\nu)}[(þ - \rho + \bar\rho)M_{\dot 2} + (ð - \tau + \bar\tau')M_{\dot 1}]\} + \text{cc} \end{aligned} \tag{7a}$$

$$b_\mu = \tfrac{1}{2}[l_\mu(ð + \tau) - m_\mu(þ + \rho)]\psi_E + \text{cc} \tag{7b}$$

respectively, with $M_{\dot 1}$, $M_{\dot 2}$, and ψ_E restricted by the equations

$$\begin{aligned} (ð' - \bar\tau)M_{\dot 2} - (þ' - \bar\rho')M_{\dot 1} &= 2\phi_1\psi_G \\ (þ + 3\rho - \bar\rho)M_{\dot 2} - (ð + 3\tau - \bar\tau')M_{\dot 1} &= 4\phi_1\psi_E \\ 2\phi_1[(ð' - \tau')\psi_E - (þ + 2\rho)\psi_G] &= (3\psi_2 + 2\phi_{11})M_{\dot 1} \\ 2\phi_1[(þ' - \rho')\psi_E - (ð + 2\tau)\psi_G] &= (3\psi_2 - 2\phi_{11})M_{\dot 2} \end{aligned} \tag{8}$$

$M_{\dot{1}}$, $M_{\dot{2}}$, ψ_G, and ψ_E have types $\{-3, 1\}$, $\{-3, -1\}$, $\{-4, 0\}$, and $\{-2, 0\}$, respectively. Since the hot RN solution is static and spherically symmetric and ψ_G, ψ_E, $M_{\dot{1}}$, and $M_{\dot{2}}$ have spin weight -2, -1, -2, and -1, respectively, we may suggest the solutions of the forms

$$\begin{aligned} \psi_G &= [N(r)/r]\ {}_{-2}Y_{lm}(\theta, \phi)\, e^{i\omega t} \\ \psi_E &= \sqrt{2}\ T(r)\ {}_{-1}Y_{lm}(\theta, \phi)\, e^{i\omega t} \\ M_{\dot{1}} &= 2et(r)\ {}_{-2}Y_{lm}(\theta, \phi)\, e^{i\omega t} \\ M_{\dot{2}} &= \sqrt{2}\ e[n(r)/r]\ {}_{-1}Y_{lm}(\theta, \phi)\, e^{i\omega t} \end{aligned} \tag{9}$$

where ${}_sY_{lm}$ are spin-weighted spherical harmonics. If we substitute equations (3)–(5) into equations (8) and use (6), we get a system of four linear ordinary differential equations:

$$\begin{aligned} \xi n + \frac{Y^2}{r^3}\left(\partial_r - i\omega\,\frac{r^2}{Y}\right)\left(\frac{r^3}{Y}\,t\right) &= \frac{N}{r} \\ r^3\left(\partial_r + i\omega\,\frac{r^2}{Y}\right)\frac{n}{r^3} + \xi t &= \frac{2T}{r} \\ -\xi T + r^3\left(\partial_r + i\omega\,\frac{r^2}{Y}\right)\frac{N}{r^3} &= 2t\left(3m - \frac{4e^2}{r}\right) \\ \frac{Y^2}{r^3}\left(\partial_r - i\omega\,\frac{r^2}{Y}\right)\left(\frac{r^3}{Y}\,T\right) - \xi N &= 2n\left(3m - \frac{2e^2}{r}\right) \end{aligned} \tag{10}$$

where $\xi = [(l-1)(l+2)]^{1/2}$ provided $l > 1$.

3. ONE-DIMENSIONAL WAVE EQUATIONS

Equations (10) can be written as

$$\begin{aligned} \frac{Y^2}{r^3}\left(\partial_r - i\omega\,\frac{r^2}{Y}\right)\left[\frac{r^3}{Y}\left(T + m_i\,\frac{t}{\xi}\right)\right] &= \xi\left(1 + \frac{m_i}{\xi^2 r}\right)\left(N + m_j\,\frac{n}{\xi}\right) \\ r^3\left(\partial_r + i\omega\,\frac{r^2}{Y}\right)\left[\frac{1}{r^3}\left(N + m_i\,\frac{n}{\xi}\right)\right] &= \xi\left(1 + \frac{2m_i}{\xi^2 r}\right)\left(T + m_j\,\frac{t}{\xi}\right) \end{aligned} \tag{11}$$

where

$$m_i + m_j = 6m, \qquad m_i m_j = -4e^2\xi^2 \qquad (i, j = 1, 2 \quad \text{and} \quad i \neq j)$$

In terms of r_* defined by

$$dr_*/dr = r^2/Y$$

the set of equations (11) reduces to

$$\begin{aligned}(d/dr+i\omega)Y_{-i} &= \xi - \frac{Y^2}{r^8}\left(1+\frac{2m_i}{\xi^2 r}\right)X_{-j}\\ (d/dr-i\omega)X_{-i} &= \xi\frac{r^4}{Y}\left(1+\frac{m_i}{\xi^2 r}\right)Y_{-j} \qquad (i\neq j)\end{aligned} \tag{12}$$

where

$$\begin{aligned}Y_{-i} &= \frac{1}{r^3}\left(N+m_i\frac{n}{\xi}\right)\\ X_{-i} &= \frac{r^3}{Y}\left(T+m_i\frac{t}{\xi}\right)\end{aligned} \tag{13}$$

Equations (12) are identical to the set of equations obtained by Chandrasekhar (1983), but with a different meaning for the unknowns. So we can follow Chandrasekhar's approach to express each pair (Y_{-i}, X_{-j}) $(i\neq j)$ in terms of a function Z_i that satisfies a Schrödinger-type wave equation and has a simple asymptotic behavior. In fact, the pair (Y_{-i}, X_{-j}) can be expressed either in terms of a function $Z_i^{(+)}$ or a function $Z_i^{(-)}$.

The function $Z_i^{(\pm)}$ satisfies

$$(d^2/dr_*^2+\omega^2)Z_i^{(\pm)} = V_i^{(\pm)}Z_i^{(\pm)} \tag{14}$$

where

$$\begin{aligned}V_i^{(\pm)} &= \pm m_j\, df_i/dr_* + m_j^2 f_i^2 + kf_i\\ f_i &= \frac{Y}{r^3(\xi^2 r+m_j)}, \qquad k=\xi^2(\xi^2+2)\end{aligned} \tag{15}$$

It can be verified that

$$\begin{aligned}Y_{-i} &= V_i^{(\pm)}Z_i^{(\pm)} + (W_i^{(\pm)}-2i\omega)(d/dr_*-i\omega)Z_i^{(\pm)}\\ \xi X_{-i} &= \mp m_i Z_i^{(\pm)} + f_i^{-1}(d/dr_*-i\omega)Z_i^{(\pm)} \qquad (i\neq j)\end{aligned} \tag{16}$$

where

$$W_i^{(\pm)} = -\frac{d}{dr_*}\ln f_i \mp m_j f_i \qquad (i\neq j) \tag{17}$$

satisfy equations (12) as a consequence of equations (14). In the RN limit, equations (14) reduce to the equations obtained by Moncrief (1974*a*,*b*; 1975) and Zerilli (1974) in their studies of the perturbations of the RN solution. Since the potentials $V_i^{(\pm)}$ are of short range, $Z_i^{(\pm)}$ have the asymptotic behaviors $e^{\pm i\omega r_*}$ both for $r_* \to \alpha$ and for $r_* \to -\alpha$. The ingoing modes have asymptotic forms

$$\overleftarrow{Z}_i^{(\pm)} \to \begin{cases} e^{i\omega r_*} + \overleftarrow{R}_i e^{-i\omega r_*}, & r_* \to \alpha \\ T_i^{(\pm)} e^{i\omega r_*}, & r_* \to -\alpha \end{cases} \tag{18}$$

with reflection and transmission coefficients R and T. The outgoing modes have the form

$$\vec{Z}_i^{(\pm)} \to \begin{cases} \vec{T}_i^{(\pm)} e^{i\omega r_*}, & r_* \to \alpha \\ e^{-i\omega r_*} + \vec{R}_i^{(\pm)} e^{i\omega r_*}, & r_* \to -\alpha \end{cases} \tag{19}$$

These modes are bounded on the horizons and therefore a general perturbation, when its mode expansion is given, is also bounded.

4. REMARKS

This result on the stability of a hot RN black hole can be applied to the inflationary scenario. We consider a situation where the radiation density is sufficiently large to overcome the cosmological expansion. This density collapses to form a black hole. Then the metric will have the form of a perturbed hot black hole spacetime and an observer sees that the spacetime approaches de Sitter spacetime.

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